Dormand-Prince method

1. Summarize adaptive algorithms.

Answer:

a. Using the current value of h, approximate the solution at $t_k + h$ using two different algorithms, one that is generally less accurate and another that is generally more accurate. Denote these approximations by y and z, respectively.
b. Overestimate the error of the approximate y using $2|y - z|$.
c. The error for a single step of the less accurate method will be of the form Ch^m for an appropriate power of m. Thus, we have that $Ch^m \approx 2|y - z|$, so $C \approx 2|y - z|/h^m$.
d. Now, if we are willing to accept an error of ε_{abs} per unit time, then we are willing to accept $h\varepsilon_{\text{abs}}$ for that interval of width h.
e. Now, if the error of y is greater than $h\varepsilon_{\text{abs}}$, the we should try again with a smaller value of h; however, if the error of y is less than $h\varepsilon_{\text{abs}}$, our step size is too small and we could have used a larger value of h. In the first case, we'll try again. In the second case, we'd like to use a larger value of h.
f. What we really want is a scalar multiple of the interval width ah that gives the maximum acceptable error. Thus, we want to find an a so that $C(ah)^m = (ah)\varepsilon_{\text{abs}}$. We can substitute in the value of C above to get that $2|y - z|/h^m \times (ah)^m = (ah)\varepsilon_{\text{abs}}$. We can solve this for a as follows:
$$a = \sqrt[m-1]{\frac{h\varepsilon_{\text{abs}}}{2|y-z|}}.$$
g. If $a \leq 1$, we should try again with $0.9ah$, but if $a > 1$, we will use z to approximate the solution at $t_k + h$ and with the next step we will use a step size of $0.9ah$.

2. What algorithm does ode45 in Matlab use?

Answer: The Dormand-Prince method.

3. What are the errors of the two approximations of the Dormand-Prince method? What is m in the above calculation of a?

Answer: For a single step, the two approximations are $O(h^5)$ and $O(h^6)$, but such algorithms are always referred to their error after multiple steps. Thus, the value of m in this case is 5, so the calculation of a is the fourth root:

$$a = \sqrt[4]{\frac{h\varepsilon_{\text{abs}}}{2|y-z|}}.$$

4. Would you ever expect to calculate even one step of the Dormand-Prince method in this course?

Answer: No. It is not unreasonable to ask for one step of the adaptive Euler-Heun method, but Dormand-Prince is an algorithm that should be implemented in a program.